Peano numerals as buck-stoppers

In this abstract certain views of Ackerman (1978) and Kripke (1992) will be examined. First, they claim that *not* any arithmetical terms is eligible for universal instantiation and existential generalisation in doxastic or epistemic contexts. Second, they claim that Peano numerals *are* eligible for universal instantiation and existential generalisation in doxastic or epistemic contexts. Third, they claim that the successor relation and the smaller-than must be effectively calculable. These three claims will be examined from the framework of modal-epistemic arithmetic, i.e. arithmetic extended with certain modal, epistemic and modal-epistemic principles. I will present theorems that give support to the second and the third claim of Ackerman and Kripke.

Let $\mathcal{L}_{\mathbf{PA}}$ denote the language of Peano Arithmetic (**PA**). Suppose one adds a *knowledge* operator $\mathbf{K}$ and a *possibility* operator $\Diamond$ to that language, yielding the language of Modal-Epistemic Arithmetic ($\mathcal{L}_{\mathbf{MEA}}$). Then one question is whether Modal-Epistemic Arithmetic (**MEA**) should contain the unrestricted principle of universal instantiation and its equivalent, the unrestricted principle of existential generalisation.

UIt $\forall x \phi \rightarrow \phi\,(t/x)$ for any $t, \phi \in \mathcal{L}_{\mathbf{MEA}}$.

EGt $\phi\,(t/x) \rightarrow \exists x \phi$ for any $t, \phi \in \mathcal{L}_{\mathbf{MEA}}$.

Ackerman (1978) and Kripke (1992) argue that the answer should be negative. (Kripke's answer was given in unpublished lectures. Steiner (2011) gives the first summary and discussion of Kripke's work on this subject in print.) To be more precise, Ackerman and Kripke rejected **EGt** when the knowledge operator is replaced by a belief operator. Moreover, Ackerman's counterexample involved an arithmetical description term, which does not belong to $\mathcal{L}_{\mathbf{PA}}$. Finally, Ackerman talks about Arabic numerals, Roman numerals and other numerals, whereas Kripke focuses on the decimal numerals. Setting aside these differences in this abstract, I will now present an Ackerman-Kripke style counterexample to **EGt**. Note that the following is an instantiation of **EGt**:

$$\mathbf{K}t = t \rightarrow \exists x \mathbf{K} x = t. \tag{1}$$

Consider the following terms t and t': $\overline{523776}$ and $\overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right)$, with $\overline{n}$ the Peano numeral for the natural number n. It is a mathematical truth that

$$\overline{523776} = \overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right).$$

Moreover, it is perfectly possible that the following is true about someone, call her Anne:

$$\mathbf{K}\overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right) = \overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right). \tag{2}$$

But it may perfectly be the case that the following is not true about Anne:

$$\exists x \mathbf{K} x = \overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right). \tag{3}$$

In order to find out whether two terms are co-denoting or to find out which number is denoted by a complex term, one needs to carry out computations. Certainly, it is possible to do so. But as long as one has not done so, one does not know. So, it would seem there is a counterexample to (1) and, therefore, **EGt**. Contrast the above with the situation in which the following is true about Anne

$$\mathbf{K}\overline{523776} = \overline{523776} \tag{4}$$

There does not seem to be a case in which it is false about Anne that (3). In Kripke's terminology, $\overline{523776}$ is a buck-stopper, whereas $\overline{2} \times \left(\overline{5} + \overline{27}\right) \times \left(\overline{8} \times \overline{1023}\right)$ is not. If Peano numerals are indeed buck-stoppers, then one accepts universal instantiation and existential generalisation restricted to Peano numerals, i.e. the constant $\mathbf{0}$ and $\mathbf{s}\left(t'\right)$ for any Peano numeral t'.

UIn $\forall x \phi \to \phi\left(t/x\right)$ for any Peano numeral t and for any $\phi \in \mathcal{L}_{\mathbf{MEA}}$.

EGn $\phi\left(t/x\right) \to \exists x \phi$ for any Peano numeral t and for any $\phi \in \mathcal{L}_{\mathbf{MEA}}$.

An obvious follow-up question is what could justify the epistemic difference between Peano numerals on the one hand and other complex arithmetical terms on the other hand. Ackerman thinks that the Peano numerals reveal the structure of the natural numbers.

> But what is special about what is expressed by numerals as compared with other standard names? I cannot answer this precisely, but, imprecisely, there seems to be a sense in which a numeral *directly* specifies the position of its referent in the progression of numbers. Of course, a standard name that is a mathematical description such as 'the smallest perfect number' also specifies (in the sense of expressing information that logically determines) the position of its referent. But the position of what a numeral refers to can be known directly simply by understanding the numeral, without having any mathematical knowledge beyond what is necessary to understand the numeral. I do not mean to suggest that one can understand '75' without having the concepts of any other numbers. '75' can be understood only in the context of a system of numbers, and knowing and understanding a system of numerals seems to be a matter of knowing how to generate in order the progression of numerals and knowing how to count transitively (e.g. to count marbles) in accord with the progression. (Ackerman, 1978, p. 151)

Saul Kripke also thought that the structure of the decimal numbers is represented in an epistemically transparent way by the decimal numerals, whereas in general this is not the case for more complex descriptions of the decimal numbers (Steiner, 2011, p. 161-166). In contrast to Ackerman, he thought that the Roman numerals denote different numbers than the numbers denoted by the decimal numerals (Steiner, 2011, p. 165). Similarly, the decimal numerals denote different numbers than the numbers denoted by Peano numerals, because the first ten decimal numerals refer to cardinal numbers rather than ordinal numbers (Steiner, 2011, p. 161-163). Kripke is aware of the problem that, when one considers very big natural numbers, representations other than the Peano numeral for that number may be more revealing (Steiner, 2011, p. 165, 175-176). This problem will be set aside as well.

Finally, Kripke stresses the importance of the effective calculability of the 'successor of' and 'smaller than' relations (Steiner, 2011, p. 163, 166). This ties in well with Ackerman's comments on the importance of 'knowing how to generate in order the progression of numerals'.

The main goal of my paper is to give support for the second and the third claim made by Ackerman and Kripke. For this purpose I will make use of a weak version of Modal-Epistemic Arithmetic, which I call $\mathbf{MEA}^{-}$. Next to **PA**, $\mathbf{EA}^{-}$ contains two restricted versions of universal instantiation and, therefore, also restricted versions of the principle of existential generalisation.

UI $\forall x \phi \to \phi(y/x)$ for all $\phi \in \mathcal{L}_{\mathbf{EA}}$.

RUIt $\forall x \phi \to \phi(t/x)$ for all $t, \phi \in \mathcal{L}_{\mathbf{PA}}$.

Principle **UI** expresses a conceptual link between quantifiers and variables. Principle **RUIt** is crucial for the full deductive power of **PA**. Furthermore, $\mathbf{EA}^{-}$ also contains the rule of universal generalisation. In addition, $\mathbf{EA}^{-}$ contains the law of self-identity and the principle of the substitutivity of identicals (both restricted to variables). The theory also contains modal system **S5**. Finally, $\mathbf{EA}^{-}$ contains the following epistemic and modal-epistemic principles.

ME1 $\mathbf{K}\phi \to \phi$.

ME2 $\vdash_{\mathbf{MEA}^{-}} \phi \quad \Rightarrow \quad \vdash_{\mathbf{MEA}^{-}} \Diamond \mathbf{K}\phi$.

ME3 $\vdash_{\mathbf{MEA}^{-}} \phi \to \psi \quad \Rightarrow \quad \vdash_{\mathbf{MEA}^{-}} \mathbf{K}\phi \to \Diamond \mathbf{K}\psi$.

A first philosophically interesting result is that **UIn** and **EGn** are theorems of $\mathbf{EA}^{-}$.

Lemma 1. $\vdash_{\mathbf{EA}^{-}} \forall x\,(x = \mathbf{0} \to (\phi \leftrightarrow \phi(\mathbf{0}/x)))$ *for all* $\phi \in \mathcal{L}_{\mathbf{MEA}}$

Proof. The proof is by the induction principle.

First, one needs to prove that

$$\mathbf{0} = \mathbf{0} \to (\phi' \leftrightarrow \phi'(\mathbf{0}/\mathbf{0})),$$

with $\phi' = \phi(\mathbf{0}/\mathbf{0})$. But this is trivial, since $\phi'(\mathbf{0}/\mathbf{0})$ is identical to ϕ'.

Second, one needs to prove that

$$\forall x\,((x = \mathbf{0} \to (\phi \leftrightarrow \phi(\mathbf{0}/x))) \to (\mathbf{s}(x) = \mathbf{0} \to (\phi' \leftrightarrow \phi'(\mathbf{0}/\mathbf{s}(x))))),$$

with ϕ' identical to $\phi(\mathbf{s}(x)/x)$. But this is trivial again, since it is an axiom of **PA** that $\forall x \mathbf{s}(x) \neq \mathbf{0}$. □

Lemma 2. $\vdash_{\mathbf{EA}^{-}} \forall x \forall y\,(x = \mathbf{s}(y) \to (\phi \leftrightarrow \phi(\mathbf{s}(y)/x)))$ *for all* $\phi \in \mathcal{L}_{\mathbf{MEA}}$ *(assuming that* $\mathbf{s}(y)$ *is substitutable for* x *in* ϕ*).*

Proof. The proof is by the induction principle.

First, one needs to prove that

$$\forall y\,(\mathbf{0} = \mathbf{s}(y) \to (\phi' \leftrightarrow \phi'(\mathbf{s}(y)/\mathbf{0}))),$$

with $\phi' = \phi(\mathbf{0}/\mathbf{0})$. This is trivial, since it is a theorem of **PA** that $\mathbf{0} \neq \mathbf{s}(y)$.

Second, one needs to prove that $\forall x$, if

$$\forall y\,(x = \mathbf{s}(y) \to (\phi \leftrightarrow \phi(\mathbf{s}(y)/x))),$$

then

$$\forall y\,(\mathbf{s}(x) = \mathbf{s}(y) \to (\phi' \leftrightarrow \phi'(\mathbf{s}(y)/\mathbf{s}(x)))),$$

with ϕ' identical to $\phi(\mathbf{s}(x)/x)$. Suppose that $\mathbf{s}(x) = \mathbf{s}(y)$. It follows by an axiom scheme of **PA** that $x = y$. Suppose that $\phi(\mathbf{s}(x)/x)$. It follows by the substitutivity of identicals that $\phi(\mathbf{s}(y)/x)$. And vice versa. □

Theorem 3. *$\vdash_{\mathbf{EA}^-} \forall x\phi \to \phi(t/x)$ for every $\phi \in \mathcal{L}_{\mathbf{MEA}}$ and for every Peano numeral t.*

Theorem 3 gives support to the proposition that Peano numerals are buck-stoppers. Futhermore, inspection of the proofs of Lemmas 1 and 2 makes it clear that the recursive axioms for the successor function and the induction principle are the driving force behind the results. The Peano numerals can be used for universal instantiation, since they are directly linked to the inductive structure of the natural numbers. By the way, no epistemic or modal-epistemic principle is used in the proofs. Therefore, **K** could also be read as a doxastic operator.

The fact that Peano numerals are buck-stoppers does not explain yet why the other arithmetical terms are not buck-stoppers. It is my conjecture that one can prove that there is an interpretation $\mathcal{M}$ of $\mathcal{L}_{\mathbf{MEA}}$ such that $\mathcal{M} \models (2)$ and $\mathcal{M} \not\models (3)$. At this moment I do not have a proof yet. (It is a complicated business to build models for $\mathcal{L}_{\mathbf{MEA}}$, cf. (Heylen, 2013)).

A second philosophically interesting result is that the successor relation and the smaller-than relation are indeed epistemically transparent. First one needs to prove that *all* arithmetical terms are eligible for universal instantiation in certain contexts.

Definition 4. Let τ be a fragment of $\mathcal{L}_{\mathbf{MEA}}$ defined as follows:

1. if ϕ is an atomic formula, then $\phi \in \tau$;
2. if ϕ is $\neg\psi$ and $\psi \in \tau$, then $\phi \in \tau$;
3. if ϕ is $\psi \to \theta$ and $\psi, \theta \in \tau$, then $\phi \in \tau$;
4. if $\phi = \forall x\psi$ and $\psi \in \tau$, then $\phi \in \tau$;
5. if $\phi = \Diamond\mathbf{K}\psi$ and $\psi \in \tau$, then $\phi \in \tau$;
6. nothing else is an element of τ.

The second main result is the following theorem.

Theorem 5. *$\vdash_{\mathbf{MEA}^-} \forall x\phi \to \phi(t/x)$ for any $t \in \mathcal{L}_{\mathbf{MEA}}$ and $\phi \in \tau$.*

In contrast to the proof of Theorem 3, epistemic and modal-epistemic principles *are* used in the proof of Theorem 5.

Next are the two desired theorems.

Theorem 6. *$\vdash_{\mathbf{MEA}^-} t = \mathbf{s}(t') \to \Diamond\mathbf{K}t = \mathbf{s}(t')$ for all $t, t' \in \mathcal{L}_{\mathbf{MEA}}$.*

Theorem 7. $\vdash_{\mathbf{MEA}^-} t < t' \to \Diamond \mathbf{K} t < t'$ *for all* $t, t' \in \mathcal{L}_{\mathbf{MEA}}$.

To sum up, Theorem 3 gives support to the Ackerman-Kripke claim that Peano numerals are buck-stoppers and Theorems 6 and 7 give support to their claim that the successor-relation and the smaller-than relation need to be effectively calculable. The proof of Theorem 3 ties in nicely with the justification given by Ackerman and Kripke for their second claim.

References

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